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Finiteness of Posets Structured by 2×2 Integer Matrices

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Finiteness of Posets Structured by 2×2 Integer Matrices

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Finiteness of Posets Structured by 2×2 Integer Matrices

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Abstract

In the late 19th century, Wilhelm Killing discovered a famous classification of the finite-dimensional complex simple Lie algebras. This result was later refined by Élie Cartan and is now referred to as Killing–Cartan classification. This result inspired many other algebraic classification results, and also manifested itself in classifications of seemingly unrelated structures. In this paper we begin an attempt to extend Killing–Cartan to encompass a certain family of ranked partially-ordered sets. We specifically focus on those posets whose structure may be described with a 2×2 integer matrix, and analyze the (in)finiteness of said posets.

1 Introduction

In 1989, A. J. Coleman, a renowned Canadian mathematician, wrote an article for the Mathematical Intelligencer titled "The Greatest Mathematical Paper of All Time" (see [Col]). While this was quite a bold claim, Coleman made a solid argument. The topic of this supposed "greatest" paper was in fact a now-famous classification of the finite-dimensional complex simple Lie algebras, which was dicovered by Wilhelm Killing in the 1880's, and later refined by Élie Cartan. At the time, Lie theory was a brand new area of mathematics, and even today, it is considered rather difficult to grasp. However, this classification kicked off over a century of efforts to classify other mathematical objects, including Wedderburn's Theorem classifying semisimple rings/algebras and the quest to classify all finite simple groups. Not only was the Killing–Cartan classification a model for these later programs, the

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techniques created and discoveries made by Killing and Cartan were useful in these other attempts.

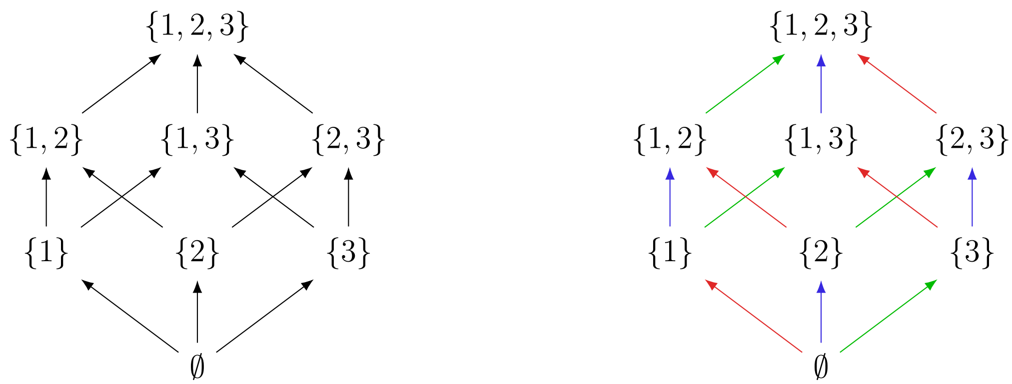
Over time, geometers, group theorists, and combinatorialists have found other manifestations of the Killing–Cartan classification: in the classification of finite root systems (see [Hum1]), in the classification of finite Coxeter groups (see [Hum2]), in classifying certain infinite-dimensional Lie algebras known as Kac–Moody alegbras (see [Kac]), and in the classification of "finitistic" numbers games (see [Erik],[DE]).

Our aim in this paper is to investigate a possibly new addition to the family of Killing–Cartan classification results. While the context for the result we seek is algebraic in several respects, the immediate environments are purely combinatorial and order-theoretic in nature, and so we investigate using discrete methods such as algorithms for constructing the objects of interest and inductive pattern-finding.

We define a *partially ordered set*, shortened to *poset*, as a set of elements S, together with an ordering, $\leq$, where the ordering is reflexive, anti-symmetric, and transitive. We write this as $P = (S, \leq)$. If $x, y \in P$, $x \leq y$, and $x \neq y$, we write $x < y$. Note that not every pair of elements has to be comparable. A *cover* of some element $x \in P$ is an element $y \in P$, where $x < y$ and there does not exist a $z \in P$ with $x < z < y$. The notation $x \to y$ denotes that y covers x. A poset is *ranked* if we can find a surjective function $\rho : P \to \{0, 1, \ldots, \ell\}$ such that $\forall x, y \in P$, if $x \to y$, then $\rho(x) + 1 = \rho(y)$. In this case we say P has length ℓ.

A canonical example of a *poset* is $P = (\mathcal{P}(\{1, 2, 3\}), \subseteq)$, where $\mathcal{P}$ is the powerset function. Not every element of P is comparable. For instance, $\{1, 2\}, \{3\} \in P$, but $\{3\} \not\subseteq \{1, 2\}$ and $\{3\} \not\supseteq \{1, 2\}$. P is also ranked: we may define $\rho : P \to \{0, 1, 2, 3\}$ by $\rho(S) = |S|$.

We often draw posets as directed graphs, where each element x corresponds to a vertex and there is an edge from x to y if y covers x. We can now draw the canonical P as in Figure 1a. Interestingly, now that we have a graphical representation of P, we may color its edges, as shown in Figure 1b. In this case, the coloring of an edge between two elements x and y is given by $x \triangle y$, where 1 is red, 2 is blue, and 3 is green.



(a) Graphical Representation

(b) Edge-Colored Graph

Figure 1: The Canonical Poset

Previously, Donnelly codified a certain structural property shared by the edge-colored and ranked posets that arise in Lie algebra and group theoretic contexts, drawing from his combinatorial study of Lie alegbra representations (see [Don1]) and symmetric functions, as well as Kashiwara's theory of "crystal graphs" and Stembridge's admissible systems (see [Stem]). (This is the so-called "M-structure," which will described in more detail below). In order to properly define this structural property, we introduce the concept of "centered coordinates." We consider chains of edges within posets and assign coordinates to each vertex along the chain. Given a chain of length ℓ, we assign the maximal vertex coordinate ℓ, the next vertex down $\ell - 2$, the next $\ell - 4$, etc. This pattern of subtracting 2 each time results in the minimal vertex along the chain having coordinate $-\ell$. Figure 2 has examples of chains of varying lengths with their appropriate coordinates.

Given a poset P structured by n colors, *with at least one edge of each color*, we assign each element $v \in P$ an n-tuple, where the ith element the tuple is the centered coordinate of v within a color i chain. If v is not part of a chain of a particular color, it is said to be part of a 0-length chain of that color and the respective coordinate is assigned a value of 0. For all colors, if the difference along any color i edge is constant throughout the entire poset, we can use these differences to construct a matrix where the ith row is the difference along any color i edge. This allows us to give the following definition:

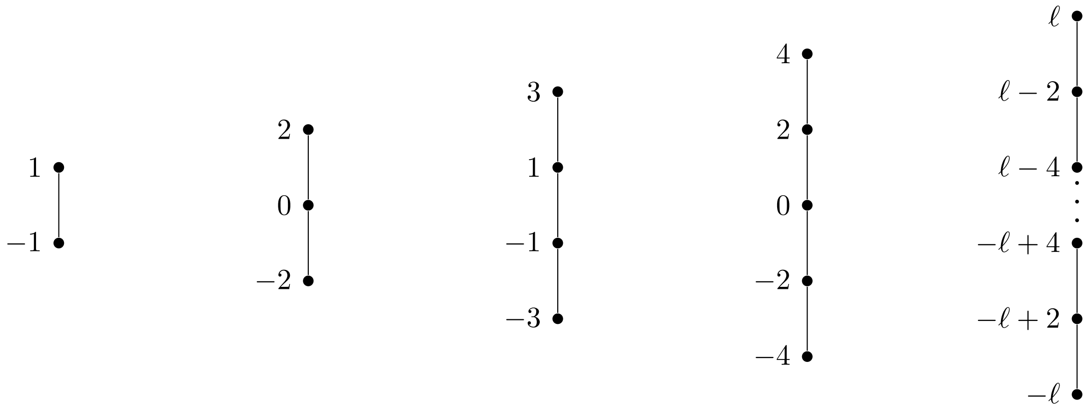


Figure 2: Centered Coordinates

Definition. A ranked poset P edge-colored by an index set $I = \{1, 2, \ldots, n\}$ is said to be *M-structured* if there exists a $n \times n$ integer matrix M such that the difference between centered coordinates along any color i edge is the ith row of M (and there is at least one edge of each color).

Our problem is to investigate 2×2 integer matrices $M = \left[\begin{smallmatrix} 2 & m \\ n & 2 \end{smallmatrix}\right]$ for which there exists a finite and M-structured poset. As indicated above, this problem is linked to, and inspired by, certain aspects of Lie theory and the study of Coxeter groups. However, the problem statement is entirely combinatorial in its flavor, and can be addressed directly using standard combinatorial reasoning; this is where we will begin our investigation. In particular, we consider the following cases:

- $mn > 0$ with both m and n negative, which has been previously analyzed by Donnelly,
- $mn = 0$ with at least one of m or n nonzero,
- $mn < 0$, and
- $mn \leq 4$ with m and n both positive.

Notably absent from this list is the case where $mn \geq 5$ with both m and n positive. It is strongly suspected that these matrices cannot structure a finite poset.

Since we are focusing on posets of 2 colors, we will assign each element an ordered pair of coordinates, (a, b), where a is the centered coordinate of a color 1 chain, and b is the centered coordinate of a color 2 chain. For the remainder of this paper, for consistency and ease of discussion, we shall say that red is color 1 and blue is color 2, and write the centered coordinate as $(a,\ b)$. We shall also cease drawing any arrows on our edges and assume all edges are directed upwards.

We will see that the number of 2×2 integer matrices admitting this particular structure property for finite posets appears to be finite, and we conjecture that the set of such matrices includes only those for which $0 < mn \leq 4$, with the latter equality only when $m = n = 2$. The case-by-case results noted above are the main contributions of this thesis; we believe the techniques used here can be extended to fully classify all 2×2 integer matrices M for which there exists a finite and M-structured poset.

2 Infinite Posets

Note. *Throughout this section, we will write "$(a,\ b) \in P$" as shorthand for "$x \in P$ with centered coordinates $(a,\ b)$".*

2.1 Setup

Lemma 2.1. *If P is a finite M-structured poset, then P must have a maximal element $(a,\ b)$ with either $a > 0$ and $b \geq 0$ or $a \geq 0$ and $b > 0$.*

Proof. Since P is finite, it must have at least one maximal element. Let $(a,\ b) \in P$ be maximal. Then $(a,\ b)$ is at the top of either a red chain or blue chain. If $(a,\ b)$ is at the top of a red chain, a must be positive. Also, $(a,\ b)$ cannot be at the bottom or in the middle of a blue chain, because then it wouldn't be maximal. Therefore it must either be at the top of a blue chain, or not in a blue chain at all, which implies b must be non-negative. So we have $a > 0$ and $b \geq 0$. If $(a,\ b)$ is at the top of a blue chain, b must be positive. Also, $(a,\ b)$ cannot be at the bottom or in the middle of a red chain, because then it wouldn't be maximal. Therefore it must either be at the top of a red chain, or not in a red chain at all, which implies a must be non-negative. So we have $a \geq 0$ and $b > 0$. $\square$

Lemma 2.2. *If P is a finite M-structured poset, then P must have a minimal element $(a,\ b)$ with either $a < 0$ and $b \leq 0$ or $a \leq 0$ and $b < 0$.*

The proof for this essentially identical to that of Lemma 2.1.

Lemma 2.3. *Let P be an M-structured poset with $M = \left[\begin{smallmatrix} 2 & m \\ n & 2 \end{smallmatrix}\right]$. If $(a,\ b) \in P$, then $(-a,\ b - am) \in P$.*

Proof. Suppose $(a,\ b) \in P$. To show $(-a,\ b - am) \in P$, there are three cases:

Case 1: If $a = 0$, $(-a,\ b - am) = (a,\ b) \in P$.

Case 2: If $a < 0$, then $(a,\ b)$ is the minimal element of a red chain of length $|a| = -a$. So we add $|a|(2,\ m)$ to $(a,\ b)$ to get the maximal element of that chain:

$$(a,\ b) + |a|(2,\ m) = (a + 2|a|,\ b + |a|m) = (-a,\ b - am).$$

Case 3: If $a > 0$, then $(a,\ b)$ is the maximal element of a red chain of length $|a| = a$. So we subtract $|a|(2,\ m)$ from $(a,\ b)$ to get the minimal element of that chain:

$$(a,\ b) - |a|(2,\ m) = (a - 2|a|,\ b - |a|m) = (-a,\ b - am).$$

□

Lemma 2.4. *Let P be an M-structured poset with $M = \left[\begin{smallmatrix} 2 & m \\ n & 2 \end{smallmatrix}\right]$. If $(a,\ b) \in P$, then $(a - bn,\ -b) \in P$.*

Proof. Suppose $(a,\ b) \in P$. Similar to above, to show $(a - bn,\ -b) \in P$, there are also three cases:

Case 1: If $b = 0$, $(a - bn,\ -b) = (a,\ b) \in P$.

Case 2: If $b < 0$, $(a,\ b)$ is the minimal element of a blue chain of length $|b| = -b$. So we add $|b|(n,\ 2)$ to $(a,\ b)$ to get the maximal element of that chain:

$$(a,\ b) + |b|(n,\ 2) = (a + |b|n,\ b + 2|b|) = (a - bn,\ -b).$$

Case 3: If $b > 0$, $(a,\ b)$ is the maximal element of a blue chain of length $|b| = b$. So we subtract $|b|(n,\ 2)$ from $(a,\ b)$ to get the minimal element of that chain:

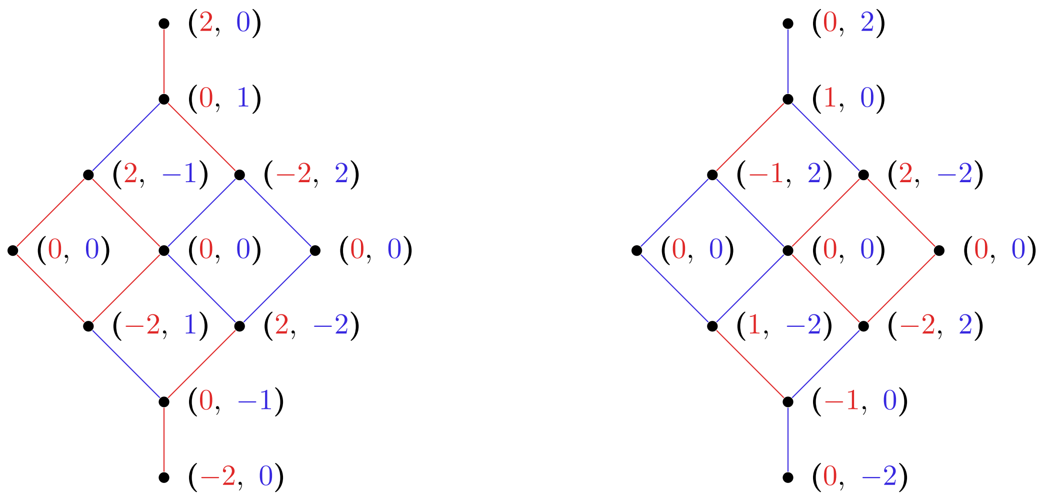
$$(a,\ b) - |b|(n,\ 2) = (a - |b|n,\ b - 2|b|) = (a - bn,\ -b).$$

$\square$

Lemma 2.5. *Let P be a poset structured by $\left[\begin{smallmatrix}2 & m\\ n & 2\end{smallmatrix}\right]$, and let P' be the poset obtained from P by swapping edge colors. Then P' is $\left[\begin{smallmatrix}2 & n\\ m & 2\end{smallmatrix}\right]$-structured. That is, when discussing finiteness, we may swap m and n without loss of generality.*

Proof. Given a $\left[\begin{smallmatrix}2 & m\\ n & 2\end{smallmatrix}\right]$-structured poset P, if we change all red edges to blue and vice versa, the coordinates of each vertex will swap from $(a,\ b)$ to $(b,\ a)$. Call this new poset P' and we have found a $\left[\begin{smallmatrix}2 & n\\ m & 2\end{smallmatrix}\right]$-structured poset in 1-1 correspondence with P. $\square$

An example of Lemma 2.5 may be seen in Figure 3.



(a) $M = \left[\begin{smallmatrix}2 & -1\\ -2 & 2\end{smallmatrix}\right]$ (b) $M = \left[\begin{smallmatrix}2 & -2\\ -1 & 2\end{smallmatrix}\right]$

Figure 3: An example of Lemma 2.5

2.2 $mn = 0$ with either m or n non-zero

Theorem 2.1. *A poset P structured by $M = \left[\begin{smallmatrix} 2 & 0 \\ n & 2 \end{smallmatrix}\right]$, $n \in \mathbb{Z}\backslash\{0\}$ has infinitely many elements.*

Proof. We will show that there exist $a, b \in \mathbb{Z} \setminus \{0\}$ such that $(a + (-1)^k kbn,\ (-1)^k b) \in P$ for all integers $k \geq 1$. We shall proceed by induction.

Base Case: $k = 1$. Since P has at least one edge of each color, we can find a blue chain of length $\ell > 0$ with maximal element $(a',\ b') \in P$. Now $b' = \ell$, and if $a' \neq 0$, choose $(a,\ b) = (a',\ b')$. If $a = 0$, then the minimal element of said blue chain is $(a' - b'n,\ -b') \in P$, with $b' = \ell$ and $a' - b'n \neq 0$, so choose $(a,\ b) = (a' - b'n,\ -b')$. Lemma 2.3 then gives that $(-a,\ b - 0 \cdot a) = (-a,\ b) \in P$, which is distinct from $(a,\ b)$ since we chose $a \neq 0$. Applying Lemma 2.4 again gives $(-a - bn,\ -b) = (-a + (-1)^1(1)bn,\ (-1)^1 b) = (-a + (-1)^k kbn,\ (-1)^k b) \in P$.

Inductive Step: Suppose $(-a + (-1)^k kbn,\ (-1)^k b) \in P$. Then applying Lemma 2.3 gives $(-(-a + (-1)^k kbn),\ (-1)^k b - 0 \cdot (-a + (-1)^k kbn)) = (a - (-1)^k kbn,\ (-1)^k b) \in P$. Applying Lemma 2.4 to that point gives $((a - (-1)^k kbn) - (-1)^k bn,\ -(-1)^k b)$
$= (a - (-1)^k (k+1)bn,\ -(-1)^k b) = (a + (-1)^{k+1}(k+1)bn,\ (-1)^{k+1} b) \in P$. $\square$

Theorem 2.2. *A poset P structured by $M = \left[\begin{smallmatrix} 2 & m \\ 0 & 2 \end{smallmatrix}\right]$, $m \in \mathbb{Z}\backslash\{0\}$ has infinitely many elements.*

Proof. By Lemma 2.5 we may find consider a poset P' structured by $M = \left[\begin{smallmatrix} 2 & 0 \\ m & 2 \end{smallmatrix}\right]$ that corresponds to P. By Theorem 2.1, P' is infinite, which implies P is infinite. $\square$

2.3 $mn < 0$

Next, we will attempt to show that if m and n have opposite signs, then a $\left[\begin{smallmatrix} 2 & m \\ n & 2 \end{smallmatrix}\right]$-structured poset P is infinite. To do this, we first explore what happens when we repeatedly apply Lemma 2.3 then Lemma 2.4 to some $(a,\ b) \in P$. The first few terms are captured in the following table (where one iteration is an application of both Lemmas, in order):

Iterations	a	b
0	$(1)a - (0)b$	$(0)a - (-1)b$
1	$(mn-1)a - (n)b$	$(m)a - (1)b$
2	$(m^2n^2 - 3mn + 1)a - (mn^2 - 2n)b$	$(m^2n - 2m)a - (mn-1)b$
3	$(m^3n^3 - 5m^2n^2 + 6mn - 1)a$ $-(m^2n^3 - 4mn^2 + 3n)b$	$(m^3n^2 - 4m^2n + 3m)a$ $-(m^2n^2 - 3mn + 1)b$
$\vdots$	$\vdots$	$\vdots$

If we let $x = mn$, then we can rewrite the table like so:

Iterations	a	b
0	$(1)a - (0)b$	$(0)a - (-1)b$
1	$(x-1)a - (n)b$	$(m)a - (1)b$
2	$(x^2 - 3x + 1)a - n(x-2)b$	$m(x-2)a - (x-1)b$
3	$(x^3 - 5x^2 + 6x - 1)a$ $-n(x^2 - 4x + 3)b$	$m(x^2 - 4x + 3n)a$ $-(x - 3x + 1)b$
$\vdots$	$\vdots$	$\vdots$

We would like to describe the behavior of the coeefficients of a and b. We consider *alternating Fibonnaci polynomials*, studied by Donnelly [Don2], so named because they are identical to the *Fibonnaci polynomials* except that the coefficients alternate signs. The polynomials are defined as follows:

$$r_k(x) = \begin{cases} -1 & k = -1 \\ 0 & k = 0 \\ x r_{k-1}(x) - r_{k-2}(x) & k > 0 \text{ and } k \text{ odd.} \\ r_{k-1}(x) - r_{k-2}(x) & k > 0 \text{ and } k \text{ even.} \end{cases}$$

For $k > 0$, $|r_k(-1)|$ is the k^{th} Fibonnaci number. The reason we define $r_{-1}(x) = -1$ shall become clear soon. The first few of these polynomials are:

k	$r_k(x)$
-1	-1
0	0
1	1
2	1
3	$x-1$
4	$x-2$
5	x^2-3x+1
6	x^2-4x+3
7	x^3-5x^2+6x-1
$\vdots$	$\vdots$

Now we may again rewrite the table of points:

Iterations	a	b
0	$r_1(x)a - r_0(x)b$	$r_0(x)a - r_{-1}(x)b$
1	$r_3(x)a - nr_2(x)b$	$mr_2(x)a - r_1(x)b$
2	$r_5(x)a - nr_4(x)b$	$mr_4(x)a - r_3(x)b$
3	$r_7(x)a - nr_6(x)b$	$mr_6(x)a - r_5(x)b$
$\vdots$	$\vdots$	$\vdots$

This pattern continues indefinitely and is easy to show by of the way we defined $r_k(x)$. Therefore, studying the properties of these polynomials should give us some insight into the behavior of our posets. This behavior may be captured as follows:

After i iterations, we have that $(r_{2i+1}(x)a - nr_{2i}(x)b,\ mr_{2i}(x)a - r_{2i-1}(x)b) \in P$.

It should be made clear that the above statement does not imply the infiniteness of any poset, merely that the elements are in the poset. If we take $m = 1$ and $n = 2$, then $x = 2$, and evaluating this pattern produces a cycle of points:

$$(a,\ b) \to (a-2b,\ a-b) \to (-a,\ -b) \to (-a+2b,\ -a+b) \to (a,\ b)$$

If we then start with $(a, b) = (2, -1)$, our cycle becomes:

$$(2, -1) \to (4, 3) \to (-2, 1) \to (-4, -3) \to (2, -1)$$

A poset containing this example can be seen in Figure 4, where the chains followed by the lemmas are darkened, and the points reached in the above cycle are larger.

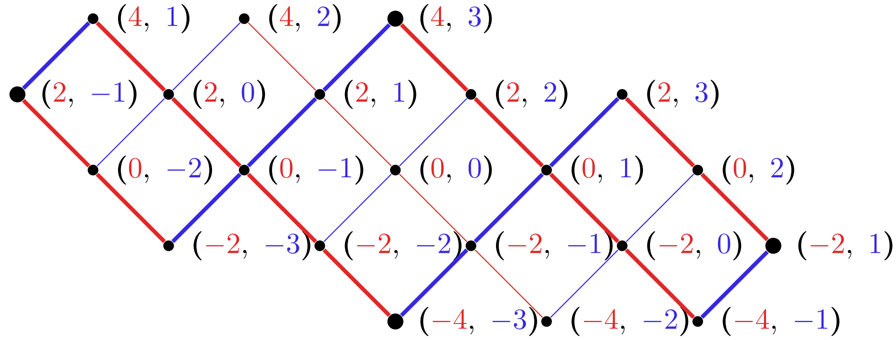


Figure 4: Cyclic example with $M = \begin{bmatrix} 2 & 2 \\ 1 & 2 \end{bmatrix}$

The following two propositions are clear and their proofs will be omitted.

Proposition 2.1. *The degree of* $r_k(x)$ *is* $\left\lfloor \frac{k+1}{2} \right\rfloor$.

Proposition 2.2. *If the degree of* $r_k(x)$ *is odd, the coefficients of the even powers of* x *are negative and those of the odd powers are positive. Similarly, if the degree of* $r_k(x)$ *is even, the coefficients of the odd powers of* x *are negative and those of the even powers are positive.*

Lemma 2.6. *If* $x < 0$ *and* $k \equiv 1 \pmod 4$ *or* $k \equiv 2 \pmod 4$, *then* $r_k(x) > 0$.

Proof. Let k be as described. Then $r_k(x)$ has degree $\left\lfloor \frac{k+1}{2} \right\rfloor$ by Proposition 2.1. If $k \equiv 1 \pmod 4$, $\left\lfloor \frac{k+1}{2} \right\rfloor = \frac{k+1}{2}$ is odd. If $k \equiv 2 \pmod 4$, $\left\lfloor \frac{k+1}{2} \right\rfloor = \frac{k}{2}$ is odd. Thus the degree of $r_k(x)$ is odd, so the odd powered terms of $r_k(x)$ have positive coefficients and the even powered terms of $r_k(x)$ have negative coefficients by Proposition 2.2. Since $x < 0$, x raised to any odd power is negative and x raised to any even power is positive. Therefore, every term of $r_k(x)$ is negative, so $r_k(x) < 0$. $\square$

Lemma 2.7. *If $x < 0$ and $k \equiv 3 \pmod 4$ or $k \equiv 4 \pmod 4 \equiv 0 \pmod 4$, then $r_k(x) < 0$.*

Proof. Let k be as described. Then $r_k(x)$ has degree $\left\lfloor \frac{k+1}{2} \right\rfloor$ by Proposition 2.1. If $k \equiv 3 \pmod 4$, $\left\lfloor \frac{k+1}{2} \right\rfloor = \frac{k+1}{2}$ is even. If $k \equiv 0 \pmod 4$, $\left\lfloor \frac{k+1}{2} \right\rfloor = \frac{k}{2}$ is even. Thus the degree of $r_k(x)$ is even, so the odd powered terms of $r_k(x)$ have negative coefficients and the even powered terms of $r_k(x)$ have positive coefficients by Proposition 2.2. Since $x < 0$, x raised to any odd power is negative and x raised to any even power is positive. Therefore, every term of $r_k(x)$ is positive, so $r_k(x) > 0$. $\square$

Lemma 2.8. *If k is odd and $x \in \mathbb{Z}$ with $x < 0$, then $|r_k(x)| > |r_{k-1}(x)|$.*

Proof. If $k \equiv 1 \pmod 4$, we have $|r_k(x)| = r_k(x)$, $|r_{k-1}(x)| = -r_{k-1}(x)$ and $r_{k-2}(x) < 0$ by Lemma 2.6 and Lemma 2.7. So then,

$$\begin{aligned} |r_k(x)| = r_k(x) &= xr_{k-1}(x) - r_{k-2}(x) \\ &> xr_{k-1}(x) \\ &\geq -r_{k-1}(x) \\ &= |r_{k-1}(x)|. \end{aligned}$$

If $k \equiv 3 \pmod 4$, we have $|r_k(x)| = -r_k(x)$, $|r_{k-1}(x)| = r_{k-1}(x)$ and $r_{k-2}(x) > 0$ by Lemma 2.6 and Lemma 2.7. So then,

$$\begin{aligned} |r_k(x)| = -r_k(x) &= -xr_{k-1}(x) + r_{k-2}(x) \\ &> -xr_{k-1}(x) \\ &\geq r_{k-1}(x) \\ &= |r_{k-1}(x)|. \end{aligned}$$

$\square$

Lemma 2.9. *If k is even and $x \in \mathbb{Z}$ with $x < 0$, then $|r_k(x)| > |r_{k-1}(x)|$.*

Proof. If $k \equiv 2 \pmod 4$, we have $|r_k(x)| = r_k(x)$, $|r_{k-1}(x)| = r_{k-1}(x)$ and $r_{k-2}(x) < 0$ by Lemma 2.6 and Lemma 2.7. So then,

$$\begin{aligned}|r_k(x)| = r_k(x) &= r_{k-1}(x) - r_{k-2}(x) \\ &> r_{k-1}(x) \\ &= |r_{k-1}(x)|.\end{aligned}$$

If $k \equiv 0 \pmod 4$, we have $|r_k(x)| = -r_k(x)$, $|r_{k-1}(x)| = -r_{k-1}(x)$ and $r_{k-2}(x) > 0$ by Lemma 2.6 and Lemma 2.7. So then,

$$\begin{aligned}|r_k(x)| = -r_k(x) &= -r_{k-1}(x) + r_{k-2}(x) \\ &> -r_{k-1}(x) \\ &= |r_{k-1}(x)|.\end{aligned}$$

□

Lemma 2.10. *In a poset P structured by $M = \left[\begin{smallmatrix} 2 & m \\ n & 2 \end{smallmatrix}\right]$, $m, n \in \mathbb{Z}$, $n > 0$, we may always find an $(a,\ b) \in P$ where a and b are either both non-negative or both non-positive.*

Proof. Let $(a,\ b) \in P$. Then we have four cases:

Case 1: $a \geq 0$ and $b \geq 0$. In this case we have found $(a,\ b)$ and are finished.

Case 2: $a \leq 0$ and $b \leq 0$. In this case we have found $(a,\ b)$ and are finished.

Case 3: $a > 0$ and $b < 0$

Then by Lemma 2.4, $(a - bn,\ -b) \in P$. Since $n, b > 0$, $a - bn < 0$ and $-b < 0$.

Case 4: $a < 0$ and $b > 0$

Then by Lemma 2.4, $(a - bn,\ -b) \in P$. Since $n > 0$ and $b < 0$, $a - bn > 0$ and $-b > 0$.

□

Theorem 2.3. *A poset P structured by $M = \left[\begin{smallmatrix} 2 & m \\ n & 2 \end{smallmatrix}\right]$, $m, n \in \mathbb{Z}$, $mn < 0$ has infinitely many elements.*

Proof. By Lemma 2.5 we may assume $m < 0$ and $n > 0$ without loss of generality. Let $x = mn$. Then by Lemma 2.10, we may find $(a,\ b) \in P$, where a and b are either both non-negative or both non-positive. To show P is infinite, we will show that the red coordinates of P increase indefinitely.

Given the above $(a,\ b)$, we have that $(r_{2i+1}(x)a - nr_{2i}(x)b,\ mr_{2i}(x)a - r_{2i-1}(x)b) \in P\ \forall i \in \mathbb{N}$. By Lemma 2.6 and Lemma 2.7, $r_{2i+1}(x)$ and $r_{2i}(x)$ have opposite signs. Since $n > 0$, $r_{2i+1}(x)$ and $-nr_{2i}(x)$ have the same sign for all $i \in \mathbb{N}$. Also, a and b have the same sign (or one of them is 0), so $r_{2i+1}(x)a$ and $-nr_{2i}(x)b$ have the same sign. So we have

$$\begin{aligned} |r_{2i+3}(x)a - nr_{2i+2}(x)b| &= |r_{2i+3}(x)a| + |nr_{2i+2}(x)b| \\ &> |r_{2i+1}(x)a| + |nr_{2i}(x)b| \text{ by Lemma 2.8 and Lemma 2.9} \\ &= |r_{2i+1}(x)a + nr_{2i}(x)b|. \end{aligned}$$

□

2.4 $mn = 4$

Theorem 2.4. *A poset P structured by $M = \left[\begin{smallmatrix} 2 & 1 \\ 4 & 2 \end{smallmatrix}\right]$ has infinitely many elements.*

Proof. By Lemma 2.10, we may find $(a,\ b) \in P$, where a and b are either both non-negative or both non-positive.

First, We will show inductively $((2k-1)a - 4(k-1)b,\ (k-1)a - (2k-3)b) \in P$ for all $k \in \mathbb{N}$.

Base case: k = 1. Then $((2k-1)a - 4(k-1)b,\ (k-1)a - (2k-3)b) =$ $(1a - 4(0)b,\ 0a - (-1)b) = (a,\ b) \in P$ by assumption.

Inductive step: Suppose $((2k-1)a - 4(k-1)b,\ (k-1)a - (2k-3)b) \in P$. Then applying Lemma 2.3 gives:

$((1-2k)a-4(1-k)b,\ (k-1)a-(2k-3)b-m[(2k-1)a-4(k-1)b])$

$=((1-2k)a-4(1-k)b,\ (-k)a-(1-2k)b) \in P.$

Now, applying Lemma 2.4 to this point gives:

$((1-2k)a-4(1-k)b-n[(-k)a-(1-2k)b],\ ka-(2k-1)b)$

$=((1-2k+4k)a-4(1-k+2k-1)b,\ ka-(2k-1)b)$

$=((2k+1)a-4kb,\ ka-(2k-1)b)$

$=((2(k+1)-1)a-4((k+1)-1)b,\ ((k+1)-1)a-(2(k+1)-3)b) \in P.$

This completes the induction. A little algebra on this point gives, $((2a-4b)k-(a-b),\ (a-2b)k-(a-3b))$. Provided $a \neq 2b$, both the red and blue coordinates of this grow indefinitely in magnitude as k gets larger. However, if $a = 2b$, then $((2k-1)a-4(k-1)b,\ (k-1)a-(2k-3)b) = (2b,\ b)$ for all k. But notice that we started at some element v_1 with coordinates $(2b,\ b)$, traveled down a red chain of length 2b, then up a blue chain of length b to get back to another element v_2 with coordinates $(2b,\ b)$. (We have that $a > 0$, which implies $b > 0$, so these chains are not of length 0). This implies that v_1 and v_2 have different ranks, and therefore must be distinct. So P has infinitely many elements. $\square$

Theorem 2.5. *A poset P structured by $M = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}$ has infinitely many elements.*

Proof. By Lemma 2.5, we may find a poset P' structured by $M = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$ that corresponds to P. By Theorem 2.4, P' is infinite, which implies P is infinite. $\square$

2.5 Conclusion

In this section, we have shown that $\left[\begin{smallmatrix} 2 & m \\ n & 2 \end{smallmatrix}\right]$-structured posets are infinite in the following cases:

- $mn = 0$ with at least one of m or n non-zero (Theorem 2.1 and Theorem 2.2)
- $mn < 0$ (Theorem 2.3)
- $m = 1$ and $n = 4$ (Theorem 2.4), and
- $m = 4$ and $n = 1$ (Theorem 2.5)

In addition, for m,n negative Donnelly has previously shown infiniteness if $mn \geq 4$ and finiteness if $0 < mn < 4$.

3 Finite Posets

Since all other $mn = 4$ cases are infinite, it seems natural to assume that any $\left[\begin{smallmatrix} 2 & 2 \\ 2 & 2 \end{smallmatrix}\right]$-structured poset would also be infinite. However, as seen in Figure 5, this is not the case. In fact, by starting with $(2k,\ 2k)$, $k \in \mathbb{N}$, it is possible to construct a finite $\left[\begin{smallmatrix} 2 & 2 \\ 2 & 2 \end{smallmatrix}\right]$-structured poset in a similar manner. However, we do note that since the red and blue differences are exactly the same, the edges are indistinguishable other than their color, and we could in fact change all red edges to be blue and vice versa, and the poset would be the same. It is also quite easy to show that the cases where $mn \in \{1, 2, 3\}$ can be finite. Examples of these can be seen in Figure 7.

Interestingly, if we relax our restrictions and allow P to be unranked, we can also build finite $\left[\begin{smallmatrix} 2 & 1 \\ 4 & 2 \end{smallmatrix}\right]$-structured posets as seen in Figure 6. This was first discovered by during the proof of Theorem 2.4 by starting with the strange $(2b,\ b)$ case and attempting to draw a poset structured by $\left[\begin{smallmatrix} 2 & 1 \\ 4 & 2 \end{smallmatrix}\right]$.

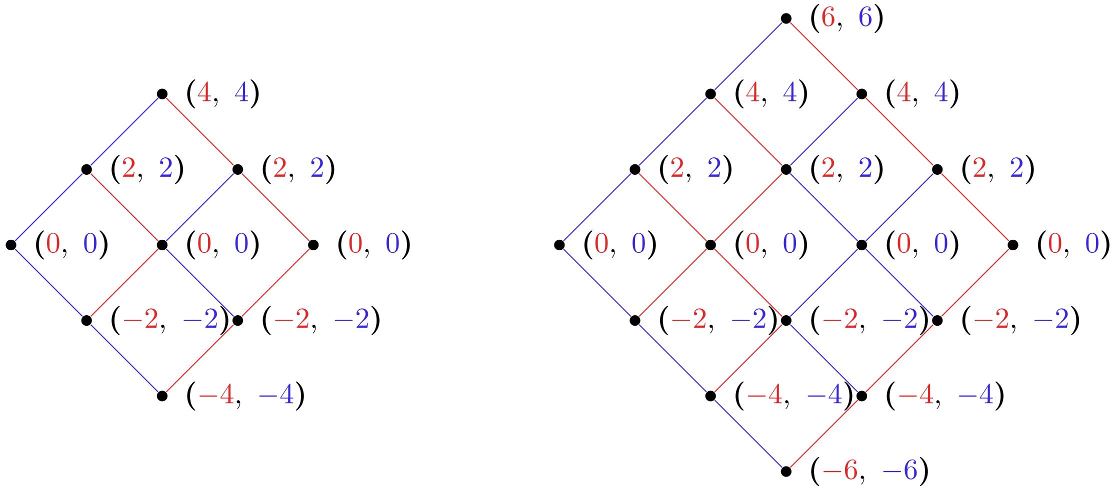


Figure 5: Finite $\left[\begin{smallmatrix} 2 & 2 \\ 2 & 2 \end{smallmatrix}\right]$-structured posets.

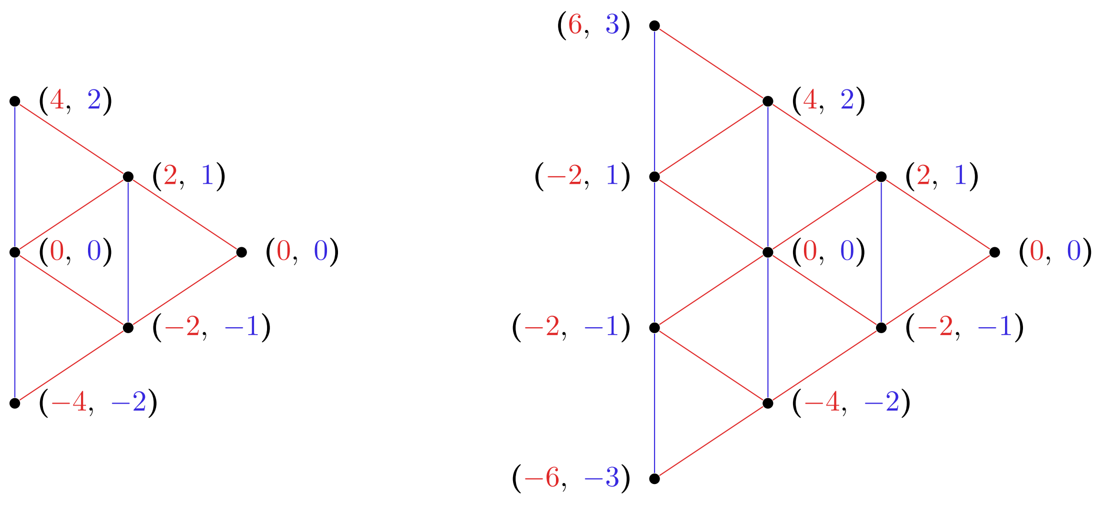


Figure 6: Unranked posets for $M = \left[\begin{smallmatrix} 2 & 1 \\ 4 & 2 \end{smallmatrix}\right]$.

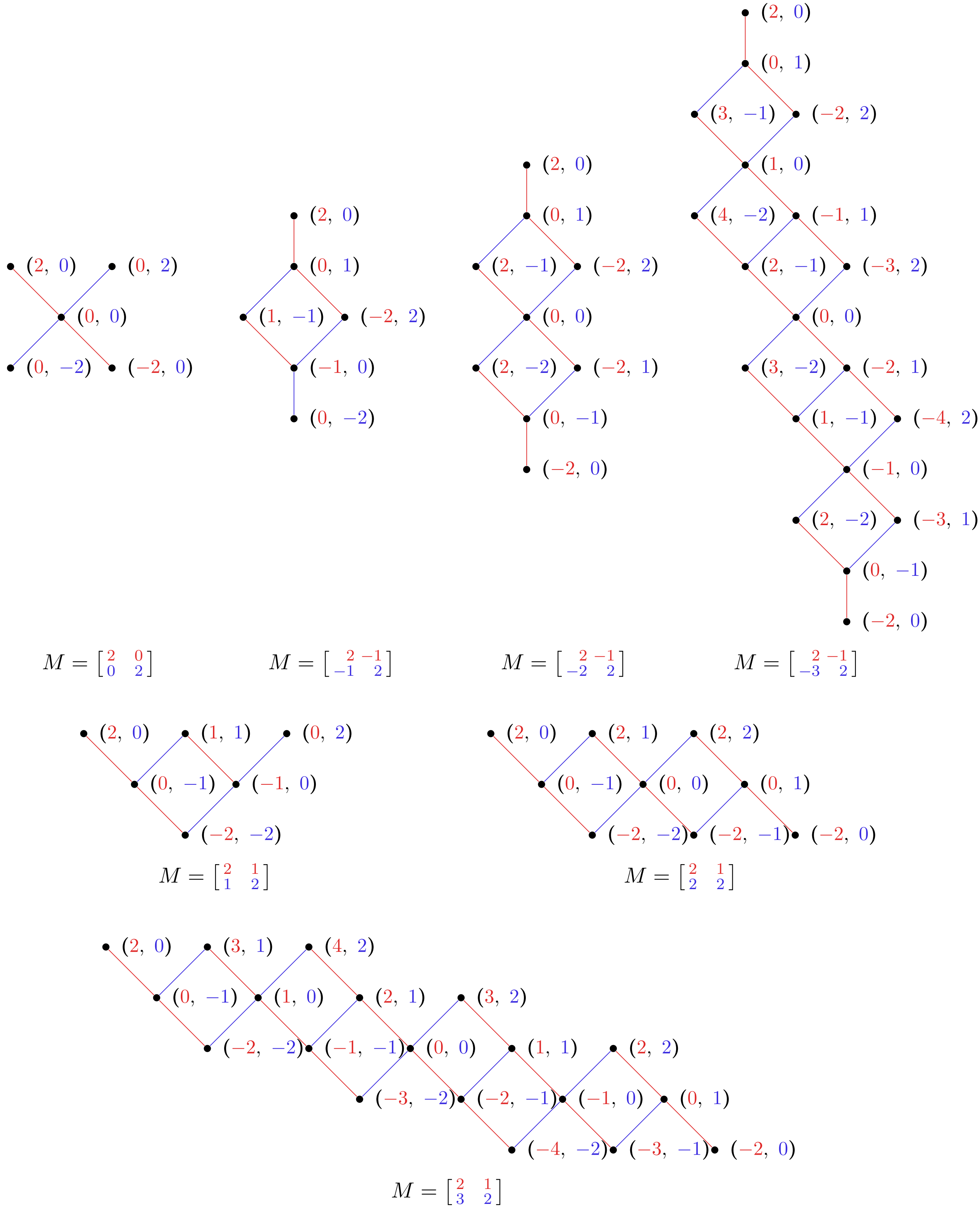


Figure 7: Examples of Finite Posets with $0 \leq mn < 4$

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